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Laplacian energy of the conjugacy class graphs of metabelian groups of order less than 30

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Abstract

Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. The conjugacy class graph $\Gamma(G)$ is defined as: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. In this paper, we compute the Laplacian energy of the conjugacy class graphs of metabelian groups of order less than thirty.

Keywords: Metabelian group, conjugacy class graph, Laplacian energy, eigenvalue.

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1. Introduction

A graph Γ is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of Γ called the edges. The vertex-set and the edge-set of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. Let Γ be a graph with vertex-set $V(\Gamma) = \{v_1, \dots, v_n\}$ and the edge-set $E(\Gamma) = \{e_1, \dots, e_m\}$. The adjacency matrix of Γ denoted by $A(\Gamma)$, is an $n \times n$ matrix defined as follows: the rows and the columns of $A(\Gamma)$ are the elements of $V(\Gamma)$. The (i, j)-entry of $A(\Gamma)$ is 0 if the vertices v_i and v_j are nonadjacent and is 1 if they are adjacent. The (i, i)-entry of $A(\Gamma)$ is 0 for $i \in \{1, \dots, n\}$. The degree of the vertex v_i is denoted by $d_\Gamma(v_i)$ and the degree matrix denoted by $\Delta(\Gamma)$ is the diagonal matrix of vertex degrees, i.e. $\Delta(\Gamma) = diag(d_\Gamma(v_1), d_\Gamma(v_2), \dots, d_\Gamma(v_n))$. The Laplacian matrix of Γ is denoted by $L(\Gamma)$ which satisfies $L(\Gamma) = \Delta(\Gamma) - A(\Gamma)$. Let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of the Laplacian matrix of Γ. The Laplacian energy of the graph Γ is defined as the sum of the absolute values of the difference between the Laplacian matrix eigenvalues and the ratio of twice the edges number divided by the vertices number, i.e., $LE(\Gamma) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, where n is the vertices number and m is the edges number of the graph Γ.
Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. From orders of representatives of conjugacy classes, the following conjugacy class graph $\Gamma(G)$ was defined in [4]: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. A metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there is an abelian normal subgroup N such that the quotient group $\frac{G}{N}$ is

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abelian. Clearly, every abelian group is metabelian. It is known that a subgroup of a metabelian group and a direct product of metabelian groups are metabelian. For further information on metabelian groups, see [3]. Recall that $H \circ K$ denotes the central product of two groups H and K, $K \rtimes H$ is the semidirect product of K and H with normal subgroup K and $K \rtimes_f H$ is the Frobenius group with kernel K and complement H. All further unexplained notations are standard. The purpose of this paper is to compute the Laplacian energy of the conjugacy class graphs of metabelian groups of order less than thirty.

2. Examples and Preliminaries

In this section, we give some examples and preliminary results that will be used in the proof of our main results.

Theorem 2.1. *Any dihedral group is metabelian.*

Proof. Suppose that $D_{2n} = \{a, b | a^n = b^2 = 1, aba = b\}$ denotes a dihedral group of order $2n$. Since the commutator subgroup is the cyclic group $\langle a^2 \rangle$, the result follows. □

Proposition 2.2. *([2]) The multiplicity of 0 as an eigenvalue of $L(\Gamma)$ is equal to the number of connected components of the graph.*

Proposition 2.3. *([1]) The Laplacian matrix of the complete graph K_n has eigenvalues 0 with multiplicity 1 and n with multiplicity $n-1$.*

Now, we give some examples of metabelian groups and find their Laplacian matrices and eigenvalues.

Example 2.4. The alternating group A_3 is an abelian normal subgroup of S_3. Since $\frac{S_3}{A_3} \cong \mathbb{Z}_2$, so the factor group of $\frac{S_3}{A_3}$ is abelian. Thus S_3 is metabelian. Also, the eigenvalue of the Laplacian matrix $\Gamma(S_3)$ is $\mu = 0$ with multiplicity 2 and we have

$$L(\Gamma(S_3)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Example 2.5. Since the center of the quaternion group Q_8 is an abelian normal subgroup of Q_8 such that $|\frac{Q_8}{Z(Q_8)}| = 4$, we deduce that Q_8 is metabelian. Also, the eigenvalues of the Laplacian matrix of $\Gamma(Q_8)$ are $\mu = 0$ with multiplicity 1 and $\mu = 3$ with multiplicity 2 and we have

$$L(\Gamma(Q_8)) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Example 2.6. Since the center of the dihedral group D_{10} is an abelian normal subgroup of D_{10} such that $|\frac{D_{10}}{Z(D_{10})}| = 5$, we deduce that D_{10} is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(D_{10})$ are $\mu = 0$ with multiplicity 2 and $\mu = 2$ with multiplicity 1 and we have

$$L(\Gamma(D_{10})) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 2.7. Since $\mathbb{Z}_3$ is an abelian normal subgroup of $Dic_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ such that the factor group of $\frac{Dic_3}{\mathbb{Z}_3}$ is abelian, Dic_3 is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(Dic_3)$ are $\mu = 0$ with multiplicity 1, $\mu = 1$ with multiplicity 1, $\mu = 3$ with multiplicity 1 and $\mu = 4$ with multiplicity 1 and we have

$$L(\Gamma(Dic_3)) = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Example 2.8. Since $\mathbb{Z}_{10}$ is an abelian normal subgroup of Dic_5 such that $|\frac{\mathrm{Dic}_5}{\mathbb{Z}_{10}}| = 2$, we deduce that Dic_5 is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(\mathrm{Dic}_5)$ are $\mu = 0$ with multiplicity 1, $\mu = 2$ with multiplicity 1, $\mu = 4$ with multiplicity 2 and $\mu = 6$ with multiplicity 2 and we have

$$L(\Gamma(\mathrm{Dic}_5)) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

Example 2.9. Since $\mathbb{Z}_8$ is an abelian normal subgroup of $M_4(2)$ such that $|\frac{M_4(2)}{\mathbb{Z}_8}| = 2$, we deduce that $M_4(2)$ is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(M_4(2))$ are $\mu = 0$ with multiplicity 1 and $\mu = 6$ with multiplicity 5 and we have

$$L(\Gamma(M_4(2))) = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}$$

Example 2.10. Since $\mathbb{Z}_2 \times \mathbb{Z}_4$ is an abelian normal subgroup of $\mathbb{Z}_4 o D_8$ such that $|\frac{\mathbb{Z}_4 o D_8}{\mathbb{Z}_2 \times \mathbb{Z}_4}| = 2$, we deduce that $\mathbb{Z}_4 o D_8$ is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(\mathbb{Z}_4 o D_8)$ are $\mu = 0$ with multiplicity 1 and $\mu = 6$ with multiplicity 5 and we have

$$L(\Gamma(\mathbb{Z}_4 o D_8)) = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}$$

3. Main results

In this section, we give our main result as follows.

Theorem 3.1. *Let G be a metabelian group of order less than 30 and $\Delta = (|g_1^G|, |g_2^G|, \ldots, |g_n^G|)$, such that g_i^G are the conjugacy classes of G for $1 \leqslant i \leqslant n$. Then the Laplacian energy of $\Gamma(G)$ is given in Table 1.*

Proof. By Theorem 2.1, the dihedral groups D_8, D_{10}, D_{12}, D_{14}, D_{16}, D_{18}, D_{20}, D_{22}, D_{24}, D_{26} and D_{28} are metabelian groups. Since the direct product of metabelian groups is metabelian, we deduce that $\mathbb{Z}_2 \times D_8$, $\mathbb{Z}_3 \times D_8$, $\mathbb{Z}_2 \times Q_8$, $\mathbb{Z}_3 \times Q_8$, $\mathbb{Z}_3 \times S_3$, $\mathbb{Z}_4 \times S_3$, $(\mathbb{Z}_2)^2 \times S_3$, $\mathbb{Z}_2 \times A_4$ and $\mathbb{Z}_2 \times \mathrm{Dic}_3$ are metabelian groups. So, the list of metabelian groups of order less than 30 be deduced from checking the groups of order less than 30 and the examples mentioned in Section 2. Now, by the orders of representatives of conjugacy classes of these groups, their Laplacian matrices and the associated eigenvalues are obtained. So, the Laplacian energy of the conjugacy class graphs of these groups can be computed by the formula $LE(\Gamma) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, as is shown in the Table 1. $\square$

Table 1: Laplacian energy of metabelian groups of order less than 30

G	Δ	Orders of representatives of conjugacy classes of G	$LE(\Gamma(G))$
S_3	$(1,3,2)$	$(1,2,3)$	0
Q_8	$(1,2,2,1,2)$	$(1,4,4,2,4)$	4
D_8	$(1,2,2,1,2)$	$(1,2,4,2,2)$	4
D_{10}	$(1,5,2,2)$	$(1,2,5,5)$	8/3
A_4	$(1,3,4,4)$	$(1,2,3,3)$	8/3
D_{12}	$(1,3,2,2,3,1)$	$(1,2,6,3,2,2)$	6
Dic_3	$(1,3,1,2,3,2)$	$(1,4,2,3,4,6)$	6
D_{14}	$(1,7,2,2,2)$	$(1,2,7,7,7)$	6
D_{16}	$(1,4,2,2,1,4,2)$	$(1,2,8,4,2,2,8)$	8
Q_{16}	$(1,4,2,2,1,4,2)$	$(1,4,8,4,2,4,8)$	8
SD_{16}	$(1,4,4,2,1,2,2)$	$(1,4,2,4,2,8,8)$	8
$M_4(2)$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,8,2,4,2,8,8,4,4,8)$	10
$\mathbb{Z}_4 o D_8$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,2,2,4,2,4,4,4,4,2)$	10
$(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_4$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,4,2,2,2,4,4,2,2,4)$	10
$\mathbb{Z}_2 \times D_8$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,2,2,2,2,4,2,2,2,4)$	10
$\mathbb{Z}_2 \times Q_8$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,4,4,2,2,4,4,4,2,4)$	10
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,4,4,2,2,4,4,4,2,4)$	10
D_{18}	$(1,9,2,2,2,2)$	$(1,2,9,3,9,9)$	48/5
$\mathbb{Z}_3 \rtimes S_3$	$(1,9,2,2,2,2)$	$(1,2,3,3,3,3)$	48/5
$\mathbb{Z}_3 \times S_3$	$(1,3,1,2,3,1,2,3,2)$	$(1,2,3,3,6,3,3,6,3)$	12
D_{20}	$(1,5,1,2,5,2,2,2)$	$(1,2,2,5,2,10,5,10)$	32/3
$\mathbb{Z}_5 \rtimes_f \mathbb{Z}_4$	$(1,5,5,4,5)$	$(1,4,2,5,4)$	6
Dic_5	$(1,5,1,2,5,2,2,2)$	$(1,4,2,5,4,10,5,10)$	32/3
$\mathbb{Z}_7 \rtimes_f \mathbb{Z}_3$	$(1,7,3,7,3)$	$(1,3,7,3,7)$	4
D_{22}	$(1,11,2,2,2,2,2)$	$(1,2,11,11,11,11,11)$	40/3
D_{24}	$(1,6,2,1,2,6,2,2,2)$	$(1,2,4,2,3,2,12,6,12)$	102/7
Dic_6	$(1,6,2,1,2,6,2,2,2)$	$(1,4,4,2,3,4,12,6,12)$	102/7
$\mathbb{Z}_3 \rtimes D_8$	$(1,6,2,1,2,6,2,2,2)$	$(1,2,2,2,3,4,6,6,6)$	102/7
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,8,4,2,3,8,8,4,12,6,8,12)$	18
$\mathbb{Z}_2 \times A_4$	$(1,1,4,3,4,3,4,4)$	$(1,2,3,2,6,2,3,6)$	32/3
$\mathbb{Z}_4 \times S_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,2,4,2,3,4,2,4,12,6,4,12)$	18
$\mathbb{Z}_3 \times D_8$	$(1,2,2,1,1,2,2,2,1,1,2,2,2,1,2)$	$(1,2,2,3,2,4,6,6,3,6,12,6,6,6,12)$	16
$(\mathbb{Z}_2)^2 \times S_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,2,2,2,3,2,2,2,6,6,2,6)$	18
$\mathbb{Z}_3 \times Q_8$	$(1,2,2,1,1,2,2,2,1,1,2,2,2,1,2)$	$(1,4,4,3,2,4,12,12,3,6,12,12,12,6,12)$	16
$\mathbb{Z}_2 \times Dic_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,4,2,2,3,4,4,2,6,6,4,6)$	18
D_{26}	$(1,13,2,2,2,2,2,2,)$	$(1,2,13,13,13,13,13,13)$	120/7
$(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$	$(1,3,3,1,3,3,3,1,3,3,3)$	$(1,3,3,3,3,3,3,3,3,3,3)$	14
$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	$(1,3,3,1,3,3,3,1,3,3,3)$	$(1,9,3,3,9,9,3,3,9,9,9)$	14
Dic_7	$(1,7,1,2,7,2,2,2,2,2)$	$(1,4,2,7,4,14,7,14,7,14)$	17
D_{28}	$(1,7,1,2,7,2,2,2,2,2)$	$(1,2,2,7,2,14,7,14,7,14)$	17

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